DEVELOPMENT OF CONVECTION IN A LIQUID LAYER CONTAINING IMPURITIES

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1. Statement of the Problem. Assume that a viscous, heat-conducting liquid containing an impurity fills an infinite, plane horizontal layer whose thickness is h. The lower boundary of the layer consists of a solid surface, which is at a constant temperature. The upper, free boundary of the layer is not deformed and is free from sheering stresses. The atmosphere above the layer consists of an immobile gas with a quasistationary temperature distribution. The vertical thermal flux Q in the atmosphere at a point remote from the free surface of the liquid is considered to be assigned (the case Q > 0 corresponds to heating from below). The temperature and the normal component of the thermal flux are continuous during passage through the free surface. There is no impurity flow through the boundaries of the layer. The liquid as a whole cannot move in a direction parallel to the bottom. The amount of impurity in the liquid does not vary with time.

The problem of determining the velocity vector $\mathbf{V} = \{v_x, v_y, v_z\}$, the pressure Π, the liquid temperature T, the atmosphere temperature Θ, and the impurity concentration S, reduced to dimensionless form and written in the Boussinesq approximation, is given by

$$\begin{gathered}\partial \mathbf{V}/\partial t + (\mathbf{V}, \nabla)\mathbf{V} = -\nabla\Pi + \Delta\mathbf{V} + \mathbf{e}(GT - G_S S),\ \partial T/\partial t + (\mathbf{V}, \nabla)T \\ = (1/P)\Delta T,\quad \operatorname{div}\mathbf{V} = 0,\quad \Delta\Theta = 0,\ \partial S/\partial t + (\mathbf{V}, \nabla)S = (1/P_d)\operatorname{div}(\nabla S \\ + \xi S\nabla T),\ \int_\Omega v_x d\Omega = \int_\Omega v_y d\Omega = \int_\Omega S d\Omega - \int_\Omega d\Omega = 0,\quad \mathbf{V} = \mathbf{0},\quad T = \partial S/\partial z \\ + \xi S\partial T/\partial z = 0\ (z = 0),\ v_z = \partial v_x/\partial z + \partial v_z/\partial x = \partial v_y/\partial z + \partial v_z/\partial y = T - \Theta \\ = \partial T/\partial z - m\partial\Theta/\partial z = \partial S/\partial z + \xi S\partial T/\partial z = 0\ \ (z = 1),\ \nabla\Theta \to \{0,\ 0, -1/m\} \\ (z \to \infty),\end{gathered} \tag{1.1}$$

where Ω is the region filled with the liquid, t is the time, $\mathbf{e} = \{0, 0, 1\}$ is the unit vector of the z axis, $G = g\beta h^4 Q/\varkappa\nu^2$ and $G_S = g\beta_S h^3 \overline{S}/\nu^2$ are the Grashof number and its concentration analog, respectively, $P = \nu/\chi$ and $P_d = \nu/d$ are the Prandtl number and its diffusion analog, respectively, $\xi = khQ/\varkappa$ is a parameter characterizing thermal diffusion, $m = \varkappa_0/\varkappa$ is the ratio of the thermal conductivity coefficient of the atmosphere $\varkappa_0$ to the thermal conductivity coefficient of the liquid $\varkappa$, g is the acceleration due to gravity, $\overline{S}$ is the mean impurity concentration under isothermic conditions, and ν, χ, β, β_S, d, and k are, respectively, the coefficients of kinematic viscosity, thermal diffusivity, thermal expansion, concentration compression, impurity diffusion, and thermal diffusion.

It should be noted that Eqs. (1.1) coincide with the ordinarily used equations for free convection of a binary mixture [1-3] if a constant value (for instance, the mean impurity concentration $\overline{S}$) is substituted for S in the term $\xi S\nabla T$. Such a substitution, however, would make equations (1.1) applicable only to the case of very small changes in the impurity concentration and would produce the impossible effect of impurity development as a result of heating.

Problem (1.1) admits of an exact solution (mechanical equilibrium):

$$\begin{gathered}\mathbf{V}_0 = \mathbf{0},\ T_0 = -z,\ S_0 = \xi\exp(\xi z)[\exp(\xi) - 1]^{-1}, \\ \Theta_0 = \frac{1 - m - z}{m},\quad \Pi_0 = \int_0^z (GT_0 - G_S S_0)\,dz + \text{const}.\end{gathered} \tag{1.2}$$

We shall now determine the secondary two-dimensional steady-state conditions originating from the equilibrium case (1.2) as the Rayleigh number R = GP passes through the critical value R_0.

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2. Two-Dimensional Steady-State Convection. Using the method described in [4], we satisfy ourselves that problem (1.1) has a two-dimensional, steady-state solution, which can be represented in the form of Lyapunov–Schmidt series:

$$\begin{gathered} v_x = \varepsilon A u_1(z) \sin \alpha x + \varepsilon^2 A^2 u_2(z) \sin 2\alpha x + \ldots, \\ v_z = \varepsilon A w_1(z) \cos \alpha x + \varepsilon^2 A^2 w_2(z) \cos 2\alpha x + \ldots, \\ T = T_0 + \varepsilon A \mathrm{P} \tau_1(z) \cos \alpha x + \varepsilon^2 A^2 \mathrm{P} [\tau_2(z) \cos 2\alpha x + \tau_0(z)] + \ldots, \\ \Theta = \Theta_0 + \varepsilon A \mathrm{P} \theta_1(z) \cos \alpha x + \varepsilon^2 A^2 \mathrm{P} [\theta_2(z) \cos 2\alpha x + \theta_0(z)] + \ldots, \\ S = S_0 + \varepsilon A \xi \mathrm{P}_d s_1(z) \cos \alpha x + \varepsilon^2 A^2 \xi \mathrm{P}_d [s_2(z) \cos 2\alpha x + s_0(z)] + \ldots, \\ \Pi = \Pi_0 + \varepsilon A q_1(z) \cos \alpha x + \varepsilon^2 A^2 [q_2(z) \cos 2\alpha x + q_0(z)] + \ldots \end{gathered} \tag{2.1}$$

Here, $\varepsilon = [(\mathrm{R} - \mathrm{R}_0)\,\mathrm{sign}\,(\mathrm{R}_0)]^{1/2}$ is a small parameter, α is the wave number of perturbations, and $A = \sqrt{|a|}$ is the amplitude of the secondary convective mode, while the constant a is found by means of the expressions

$$\begin{gathered} a = \frac{4I_1}{\mathrm{R}_0 I_2}, \quad I_1 = \int\limits_0^1 [(D^2 w_1)^2 + 2\alpha^3 (D w_1)^2 + \alpha^4 w_1^2]\, dz, \\ I_2 = \int\limits_0^1 [f_1 w_1 + \alpha^2 \mathrm{R}_0 \mathrm{P} (1 + \mu\delta) f_2 \tau_1 + \alpha^2 \mathrm{R}_0 \mathrm{P}_d \mu f_3 (s_1 + \delta\tau_1)]\, dz, \\ f_1 = 3\alpha^2 (w_1 D w_2 + 2 w_2 D w_1) - w_1 D^3 w_2 - 2 D w_1 D^2 w_2 + D w_2 D^2 w_1 + \\ + 2 w_2 D^3 w_1, \; f_2 = \tau_1 D w_2 + 4\tau_2 D w_1 + 2 w_2 D\tau_1 + 2 w_1 (D\tau_2 + 2D\tau_0), \\ f_3 = s_1 D w_2 + 4 s_2 D w_1 + 2 w_2 D s_1 + 2 w_1 (D s_2 + 2 D s_0), \end{gathered} \tag{2.2}$$

where $\mu = \beta_S k \chi \bar{S}/\beta_d$ is the thermal concentration parameter, $D = d/dz$; $\delta = \mathrm{P}/\mathrm{P}_d$.

In deriving expressions (2.2), we assumed that $\xi = khQ/\varkappa \approx 0$. We thereby limit ourselves to the case of a not too thick layer and liquids where the thermal diffusion coefficient k is small in comparison with the thermal conductivity coefficient $\varkappa$. It is, of course, not difficult to consider also the case $\xi \neq 0$; however, expressions (2.2) then become too cumbersome.

In order to find the critical value R_0 and the functions w_1, τ_1, and s_1, it is necessary to find the "first" (smallest with respect to absolute value) eigenvalue and the corresponding eigensolution of the spectral problem,

$$\begin{gathered} (D^2 - \alpha^2)^2 w_1 = \alpha^2 \mathrm{R}_0 [(1 + \mu\delta)\tau_1 - \mu\psi_1], \\ (D^2 - \alpha^2)\tau_1 = -w_1, \; (D^2 - \alpha^2)\psi_1 = w_1, \; \psi_1 = s_1 + \delta\tau_1, \\ w_1 = D w_1 = \tau_1 = D\psi_1 = 0 \;\; (z = 0), \\ w_1 = D^2 w_1 = D\tau_1 + m\alpha\tau_1 = D\psi_1 = 0 \;\; (z = 1). \end{gathered} \tag{2.3}$$

The functions w_2, τ_2, and s_2 are found by solving the nonhomogeneous boundary-value problem

$$\begin{gathered} (D^2 - 4\alpha^2)^2 w_2 = 4\alpha^2 \mathrm{R}_0 [(1 + \mu\delta)\tau_2 - \mu\psi_2] + \\ + w_1 D^3 w_1 - D w_1 D^2 w_1, \; \psi_2 = s_2 + \delta\tau_2, \\ (D^2 - 4\alpha^2)\tau_2 = -w_2 + \mathrm{P}(w_1 D\tau_1 - \tau_1 D w_1)/2, \\ (D^2 - 4\alpha^2)\psi_2 = w_2 + \mathrm{P}_d (w_1 D s_1 - s_1 D w_1)/2, \\ w_2 = D w_2 = \tau_2 = D\psi_2 = 0 \;\; (z = 0), \\ w_2 = D^2 w_2 = D\tau_2 + 2m\alpha\tau_2 = D\psi_2 = 0 \;\; (z = 1). \end{gathered} \tag{2.4}$$

The other coefficients of expansion (2.1) are found by means of the expressions

$$\begin{gathered} u_1 = -D w_1/\alpha, \; u_2 = -D w_2/2\alpha, \; \theta_0 = \tau_0(1), \\ \theta_1 = \tau_1(1) \exp [\alpha(1 - z)], \; \theta_2 = \tau_2(1) \exp [2\alpha(1 - z)], \\ q_0 = \mathrm{R}_0 \int\limits_0^z (\tau_0 - \mu s_0)\, dy, \quad q_1 = \frac{1}{\alpha^2} D^3 w_1 - D w_1, \end{gathered}$$

TABLE 1

μ	α_*	R_*	A	σ_2
4	0,567	80	$4{,}31\cdot10^{-5}$	$5{,}13\cdot10^{-6}$
3	0,815	105	$4{,}51\cdot10^{-5}$	$8{,}45\cdot10^{-6}$
2	1,090	150	$4{,}44\cdot10^{-5}$	$1{,}15\cdot10^{-5}$
1	1,455	256	$4{,}06\cdot10^{-5}$	$1{,}54\cdot10^{-5}$
0	2,100	681	$2{,}53\cdot10^{-3}$	$1{,}49\cdot10^{-3}$

$$q_2 = [D^3w_2 - 4\alpha^2Dw_2 - w_1D^2w_1 + (Dw_1)^2]/4\alpha^2,$$

$$\tau_0 = \frac{P}{2}\int\limits_0^z w_1\tau_1 dy, \quad s_0 = \varphi_0 - \int\limits_0^1 \varphi_0 dy, \quad \varphi_0 = \frac{P_d}{2}\int\limits_0^z w_1(s_1 - \delta^2\tau_1)\, dy.$$

In accordance with [5], we satisfy ourselves that the decrements σ and σ' of the equilibrium case (1.2) and the secondary mode (2.1) can be expanded in series derived from the perturbation theory:

$$\sigma = \sigma_2(R - R_0) + \sigma_4(R - R_0)^2 + \ldots, \sigma' = \sigma_2'\varepsilon^2 + \sigma_4'\varepsilon^4 + \ldots \tag{2.5}$$

The leading coefficients of series (2.5) are related by the expressions

$$\sigma_2' = -2\sigma_2\,\mathrm{sign}\,(R_0), \quad \sigma_2 = I_1/R_0I_3,$$

$$I_3 = \int\limits_0^1 [(Dw_1)^2 + \alpha^2w_1^2]\,dz + \alpha^2R_0P_d\int\limits_0^1 [\mu s_1^2 + \mu\delta s_1\tau_1 + \delta(1 + \mu\delta)\,\tau_1^2]\,dz.$$

Assume that R_0 is the single eigenvalue of problem (2.3) and that $aR_0 > 0$ and $\sigma_2R_0 > 0$. It then follows from the results obtained in [4, 5] that the equilibrium (1.2) is stable for small subcriticalities ($|R| < |R_0|$) and unstable for small supercriticalities ($|R| > |R_0|$). As R passes through the critical value R_0, there develops smoothly the secondary steady-state mode, which is represented in the form of convergent series (2.1) and is uniquely determined (with an accuracy to the shift along the x axis) by the wave number α for fixed values of μ, m, P, and P_d. This secondary mode is stable for small supercriticalities with regard to infinitesimal two-dimensional perturbations having the same periodicity and parity with respect to x as the functions (2.1).

The signs of R_0, a, and σ_2 are determined by calculating these values by means of a computer; however, in certain particular cases, this can be done without resorting to calculations. Using the method described in [4, 5], we satisfy ourselves that the following statements hold. If $\mu \geqslant 0$, then $R_0 > 0$ and $\sigma_2 > 0$. If $3\mu\delta + 4 \leqslant 0$, then $R_0 < 0$ and $\sigma_2 < 0$. If $\mu \geqslant 0$ and $\delta = 0$, then $a > 0$.

3. Numerical Results. The spectral problem (2.3) and the nonhomogeneous problem (2.4) were solved numerically by means of a BÉSM-4 computer, using the ranging method. In calculations, we performed numerical minimization of the critical value $R_0(\alpha)$ with respect to α, i.e., we sought the value α_* of the wave number α corresponding to the most critical perturbations. The eigensolution of problem (2.3) was normalized by using the condition $\tau_1(1) = 1$, which imparted a simple physical meaning to the amplitude of the secondary convective mode A: The dimensionless temperature at the free surface of the layer obeys the law

$$T = -1 + \varepsilon AP\cos\alpha x + O(\varepsilon^2).$$

The calculations were performed for the case P = 7, P_d = 813, m = 0.0436, and $\mu \geqslant 0$ corresponding to a sea water layer (salt is the impurity in the liquid) whose free surface is in contact with air. Table 1 provides the results obtained in calculating the critical value of the wave number α_*, the critical value R_* minimized with respect to α, the amplitude of the secondary convective mode A, and the leading coefficient σ_2 in the expansion of the decrement σ of the equilibrium (1.2) in series (2.5) of the perturbation theory as functions of the thermal concentration parameter μ. For the parameter values considered in this problem, the values of R_*, a, and σ_2 were positive for any $\mu \geqslant 0$. This indicates that the two-dimensional secondary steady-state mode is excited softly in a sea water layer and is stable for small supercriticalities.

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CALCULATION OF NONSTATIONARY MIXED CONVECTION OF BINARY GAS MIXTURES IN THE PRESENCE OF LARGE DENSITY VARIATIONS

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Nonstationary mixed-convective flows of gas and gas mixtures are extremely widespread in nature and technology. Their study is necessary, for example, for developing safe methods for handling toxic and explosive mixtures, solving a number of ecological problems, and industrial hygiene. In spite of the considerably subsonic nature of such flows, the spatial–temporal variation of the density in the flow, due to the nonisothermality or difference in the molecular weights of the components of the mixture, can in many cases be very significant. Nevertheless, until recently, the theoretical analysis of mixed-convective flows, just as the solution of the problems of natural convection, was based primarily on the use of the so-called Boussinesq approximation [1], which is based on the assumption that the density variations in the flow are small. In [2, 3] a system of equations is formulated, which, in contrast to the Boussinesq approximation, can be used to describe the convection of binary gas mixtures in the presence of arbitrary finite variations of the density, which greatly expanded the possibilities of numerical modeling of such flows.

In this paper, the approach adopted in [2, 3] is generalized to the case of mixed-convective flows.

The basic difference between the derivation, proposed below, of the approximate system of equations of mixed convection and the analogous derivation of the system of equations of natural convection, described in [2, 3], lies in the choice of scales used to put the complete system of Navier–Stokes equations, on which the analysis is based, into dimensionless form. This difference is due to the appearance of an additional dimensional parameter – the characteristic velocity of forced convection – in problems of mixed convection. To illustrate the choice of scales, we shall examine the following problem. Let a region, with the shape of a rectangular parallelepiped, be filled with gas with molecular weight m_2 at a temperature T_2. Initially, another gas, whose molecular weight is m_1 and whose temperature is T_1 (for definiteness $T_2 > T_1$, $m_2 > m_1$), begins to enter the volume with velocity v_1 through the opening ef (Fig. 1). Simultaneously, the same gas that filled the volume initially is introduced into the region with velocity v_2 through the opening *a*b. The mixture formed flows out of the volume through the opening cd. The problem is to calculate the development of the velocity, concentration, and temperature fields of the mixture in the volume as a function of time.

The problem described above is, on the one hand, quite typical for the class of flows under examination and, on the other, it is of certain practical interest, because it models the situation arising with activation of emergency forced exhaust ventilation when a foreign gas begins to enter an enclosure.

To put the system of Navier–Stokes equations, which describes the flow under study, into dimensionless form, we shall select as scales the following characteristic parameters of the problem: the average geometric value of the characteristic velocities of forced and natural convection $v_0 = \sqrt{v_2(gL_2\varepsilon_1)^{1/2}}$ as the velocity scale [here $\varepsilon_1 = (m_2/m_1 - 1)(T_2/T_1 - 1)$ in the case of nonisothermal flow of the gas mixture, $\varepsilon_1 = \varepsilon_T = T_2/T_1 - 1$ in the case of nonisothermal

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